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A Jenkins–Serrin problem on the strip

M. Magdalena Rodríguez

Laboratoire d'Analyse et de Mathématiques Appliquées, Université de Marne-la-Vallée, 5, Bd Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée, cedex 2, France

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Abstract

We describe the family of minimal graphs on strips with boundary values $\pm\infty$ disposed alternately on edges of length 1, and whose conjugate graphs are contained in horizontal slabs of width 1 in $\mathbb{R}^3$. We can obtain as limits of such graphs the helicoid, all the doubly periodic Scherk minimal surfaces and the singly periodic Scherk minimal surface of angle $\pi/2$.

MSC: 49Q05; 53A10

Keywords: Minimal graph; Periodic minimal surface; Jenkins–Serrin problem; Toroidal half-plane layers; KMR examples

1. Introduction

Karcher [2,3] constructed a class of doubly periodic minimal surfaces, called *toroidal half-plane layers*, from minimal graphs, by extending such graphs by symmetries. More precisely, he considered the solution to the minimal graph equation on a rectangle with boundary values 0 on the longer edges and $+\infty$ on the shorter ones; and he extended such a minimal graph to a whole strip by rotating it by an angle π about the straight segments corresponding to the boundary values 0 (see the upper picture in Fig. 2). The toroidal half-plane layer is obtained from this Jenkins–Serrin graph on the strip by considering the π-rotation about the vertical straight lines on its boundary. Such a doubly periodic example is denoted by $M_{\theta,\frac{\pi}{2},\frac{\pi}{2}}$ in [7]. Indeed, this is a particular case in the three-parametric family of KMR examples $M_{\theta,\alpha,\beta}$, with $\theta \in (0, \frac{\pi}{2})$, $\alpha \in (\frac{-\pi}{2}, \frac{\pi}{2}]$, $\beta \in [0, \pi)$ and $(\alpha, \beta) \neq (0, \theta)$, examples which have been classified in [6] as the only properly embedded, doubly periodic minimal surfaces with parallel ends and genus 1 in the quotient. Similarly to the construction of $M_{\theta,\frac{\pi}{2},\frac{\pi}{2}}$, Karcher obtained the KMR example $M_{\theta,0,\frac{\pi}{2}}$ by considering the solution to the Jenkins–Serrin problem on a rectangle with boundary values 0 on its longer edges and $+\infty$, $-\infty$ on its shorter ones (see Fig. 2, bottom). He also described a continuous deformation from $M_{\theta,\frac{\pi}{2},\frac{\pi}{2}}$ to $M_{\theta,0,\frac{\pi}{2}}$, which corresponds to the surfaces denoted by $M_{\theta,\alpha,\frac{\pi}{2}}$ in [7], with $\alpha \in [0, \frac{\pi}{2}]$, and pointed out that the intermediate surfaces did not have enough symmetries for constructing them as Jenkins–Serrin graphs.

We prove that it is possible to construct each $M_{\theta,\alpha,\frac{\pi}{2}}$, with $\alpha \in (\frac{-\pi}{2}, \frac{\pi}{2}]$, from a Jenkins–Serrin graph on a parallelogram $\mathcal{P}$ with boundary values $+\infty$ on its shorter edges and bounded data f_1, f_2 on its longer ones, and this

E-mail address: Magdalena.Rodriguez@univ-mlv.fr.

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graph can be extended to a Jenkins–Serrin graph on the strip (see the middle picture in Fig. 2). In this case, such an extension does not consist of a rotation about a straight line, but of the composition of the reflection symmetry across the plane containing the parallelogram $\mathcal{P}$ and the translation by the shorter edges on $\partial\mathcal{P}$. In particular, it must hold that $f_1 = -f_2$. Recently, Mazet [4] has constructed, in a theoretical way, these Jenkins–Serrin graphs on the strip.

Given $h > 0$ and $a \in (\frac{-1}{2}, \frac{1}{2}]$, consider $p_n = (n - a, 0, -h)$ and $q_n = (n + a, 0, h)$, for every $n \in \mathbb{Z}$. We define the strip $S(h, a) = \{(x_1, 0, x_3) \mid -h < x_3 < h\}$ and mark its boundary straight lines by $+\infty$ on the straight segments (p_{2k}, p_{2k+1}), (q_{2k}, q_{2k+1}) and $-\infty$ on (p_{2k-1}, p_{2k}), (q_{2k-1}, q_{2k}). Note that we do not consider $S(h, \frac{-1}{2})$ because it coincides with $S(h, \frac{1}{2})$.

Definition 1. We will say that a minimal graph defined on $S(h, a)$ *solves the Jenkins–Serrin problem on* $S(h, a)$ if its boundary values are $\pm\infty$ as prescribed above on each unitary segment (p_n, p_{n+1}), $(q_n, q_{n+1}) \subset \partial S(h, a)$.

We know from [1] that, in order to solve the Jenkins–Serrin problem on $S(h, a)$, it must be satisfied that $|q_0 - p_0| > 1$; this is, $a^2 + h^2 > \frac{1}{4}$. We define the collection of marked strips

$$\mathcal{S} = \left\{ S(h, a) \mid h > 0 \text{ and } a \in \left(\frac{-1}{2}, \frac{1}{2} \right] \text{ satisfy } a^2 + h^2 > \frac{1}{4} \right\}.$$

Theorem 1. *For every marked strip $S(h, a) \in \mathcal{S}$, there exist $\theta \in (0, \frac{\pi}{2})$ and $\alpha \in (\frac{-\pi}{2}, \frac{\pi}{2}]$ such that a piece of the KMR example $M_{\theta,\alpha,\frac{\pi}{2}}$ solves the Jenkins–Serrin problem on $S(h, a)$. Moreover, if a minimal graph M solves the Jenkins–Serrin problem on some $S(h, a) \in \mathcal{S}$ and its conjugate surface is contained in the slab $\{(x_1, x_2, x_3) \mid 0 < x_2 < 1\}$ up to a translation, then M must be a piece of a KMR example $M_{\theta,\alpha,\frac{\pi}{2}}$.*

2. The KMR examples $M_{\theta,\alpha,\frac{\pi}{2}}$

We know [6] that the space of doubly periodic minimal surfaces in $\mathbb{R}^3$ with parallel ends and genus 1 in the quotient coincides with the family of KMR examples $\{M_{\theta,\alpha,\beta} \mid \theta \in (0, \frac{\pi}{2}), \alpha \in (\frac{-\pi}{2}, \frac{\pi}{2}], \beta \in [0, \pi), (\alpha, \beta) \neq (0, \theta)\}$, which has been studied in detail and classified in [7] (we will keep the notation introduced there). We do not consider the example $M_{\theta,\frac{-\pi}{2},\beta}$ because it coincides with $M_{\theta,\frac{\pi}{2},\beta}$, for every θ, β. Here we sketch some properties of the subfamily $\{M_{\theta,\alpha,\frac{\pi}{2}}\}_{\theta,\alpha}$.

Given $\theta \in (0, \frac{\pi}{2})$ and $\alpha \in [0, \frac{\pi}{2}]$, the minimal surface $M_{\theta,\alpha,\frac{\pi}{2}}$ is determined by the Weierstrass data

$$g(z, w) = -\mathrm{i} + \frac{2}{\mathrm{e}^{\mathrm{i}\alpha} z - \mathrm{i}} \quad \text{and} \quad \mathrm{d}h = \mu \frac{\mathrm{d}z}{w}, \quad \mu \in \mathbb{R} - \{0\}$$

(here g is the Gauss map of $M_{\theta,\alpha,\frac{\pi}{2}}$ and $\mathrm{d}h$ is its height differential), defined on the rectangular torus $\Sigma_\theta = \{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z^2 + \lambda_\theta^2)(z^2 + \lambda_\theta^{-2})\}$, where $\lambda_\theta = \cot \frac{\theta}{2}$. The ends of $M_{\theta,\alpha,\frac{\pi}{2}}$, which are horizontal and of Scherk type, correspond to the zeros A', A''' and poles A, A'' of g (i.e. those points with $z = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}$ and $z = \mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}$, respectively). And the Gauss map g of $M_{\theta,\alpha,\frac{\pi}{2}}$ has four branch points on Σ_θ: $D = (-\mathrm{i}\lambda_\theta, 0)$, $D' = (\mathrm{i}\lambda_\theta, 0)$, $D'' = (\frac{\mathrm{i}}{\lambda_\theta}, 0)$ and $D''' = (\frac{-\mathrm{i}}{\lambda_\theta}, 0)$.

The multivalued, doubly periodic map $z : \Sigma_\theta \to \overline{\mathbb{C}}$ is used in [7] to describe a conformal model of Σ_θ as a quotient of the plane by two orthogonal translations l_1, l_2. One of the advantages is that we can read directly the z-values in this model. A fundamental domain in $\mathbb{C}$ of the action of the group generated by l_1, l_2 is the parallelogram $\widetilde{\Sigma}_\theta$ represented in Fig. 1. Each vertical line on $\widetilde{\Sigma}_\theta$ corresponds to a horizontal level section of $M_{\theta,\alpha,\frac{\pi}{2}}$ (i.e. a set x_3^{-1} (constant), where $x_3 = \Re \int \mathrm{d}h$ on $M_{\theta,\alpha,\frac{\pi}{2}}$). The curve γ drawn in Fig. 1 represents a homology class in $\Sigma_\theta - \{A, A', A'', A'''\}$ with vanishing period. Since the periods of $M_{\theta,\alpha,\frac{\pi}{2}}$ at its ends are

$$\mathrm{Per}_A = \mathrm{Per}_{A'} = -\mathrm{Per}_{A''} = -\mathrm{Per}_{A'''} = \left(\mu \frac{\pi \sin\theta}{\sqrt{1 - \sin^2\theta \cos^2\alpha}}, 0, 0 \right),$$

we conclude that every vertical line in $\widetilde{\Sigma}_\theta$ corresponds to a curve in Σ_θ with period $\pm\mathrm{Per}_A$. We fix μ so that $P = \mathrm{Per}_A = (2, 0, 0)$.

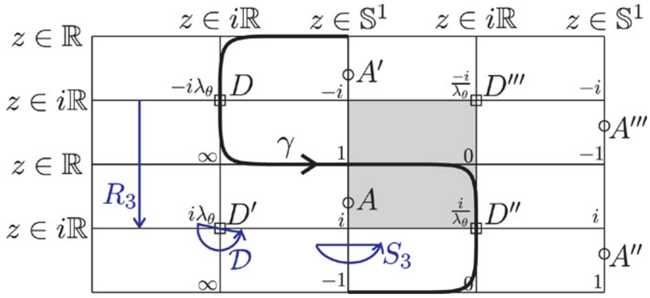

Fig. 1. The torus $\widetilde{\Sigma}_\theta$. The value appearing at each intersection point of a horizontal and a vertical line refers to the value of the z-map at the corresponding point.

The flux vectors of $M_{\theta,\alpha,\frac{\pi}{2}}$ at its ends are $\mathrm{Fl}_A = -\mathrm{Fl}_{A'} = -\mathrm{Fl}_{A''} = \mathrm{Fl}_{A'''} = (0, -2, 0)$. Thus we say that A, A''' (resp. A', A'') are *left ends* (resp. *right ends*).

If we denote by $\widetilde{\gamma} \subset \Sigma_\theta$ the curve which corresponds in $\widetilde{\Sigma}_\theta$ to the horizontal line passing through D, D''', then the flux of $M_{\theta,\alpha,\frac{\pi}{2}}$ along $\widetilde{\gamma}$ equals $-\mathrm{Fl}_A$, and the period of $M_{\theta,\alpha,\frac{\pi}{2}}$ along $\widetilde{\gamma}$ can be written as $T = (T_1, 0, T_3)$, with $T_3 \neq 0$. In particular, T is never horizontal, and $M_{\theta,\alpha,\frac{\pi}{2}}$ is a doubly periodic minimal surface with period lattice generated by P, T.

For every $\theta \in (0, \frac{\pi}{2})$ and $\alpha \in [0, \frac{\pi}{2}]$, we can similarly define the surface $M_{\theta,-\alpha,\frac{\pi}{2}}$ which coincides with the reflected image of $M_{\theta,\alpha,\frac{\pi}{2}}$ with respect to a plane orthogonal to the x_1-axis. Finally, recall from [7] that the conjugate surface of $M_{\theta,\alpha,\frac{\pi}{2}}$ coincides (up to normalization) with the KMR example $M_{\frac{\pi}{2}-\theta,\alpha,0}$, and its periods (resp. flux vectors) at the ends point to the x_2-direction (resp. x_1-direction).

2.1. Isometries of $M_{\theta,\alpha,\frac{\pi}{2}}$

The surface $M_{\theta,\alpha,\frac{\pi}{2}}$ has four horizontal straight lines traveling from left to right ends. The π-rotation about any of those straight lines induces the same isometry S_3 of $M_{\theta,\alpha,\frac{\pi}{2}}$, which corresponds to a symmetry of $\widetilde{\Sigma}_\theta$ across any of the two vertical lines passing through the ends.

Another isometry of $M_{\theta,\alpha,\frac{\pi}{2}}$, denoted by $\mathcal{D}$, is induced by the deck transformation, and corresponds to the central symmetry across any of the four branch points of g in either $\mathbb{R}^3$ or $\widetilde{\Sigma}_\theta$.

The isometry group of $M_{\theta,\alpha,\frac{\pi}{2}}$, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, is generated by $S_3, \mathcal{D}$ and R_3, where R_3 corresponds to the composition of a reflection symmetry across the plane orthogonal to the x_2-axis containing the four branch points of g, with a translation by $(1, 0, 0)$. The isometry R_3 corresponds in $\widetilde{\Sigma}_\theta$ to the translation by half a vertical period; see Fig. 1.

When $\alpha = 0, \pi/2$, the isometry group of $M_{\theta,\alpha,\frac{\pi}{2}}$ is richer (it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$), but we will not use this fact in this work. This is the lack of isometries that Karcher referred to for the intermediate surfaces $M_{\theta,\alpha,\frac{\pi}{2}}, 0 < \alpha < \frac{\pi}{2}$.

3. $M_{\theta,\alpha,\frac{\pi}{2}}$ as a graph over the x_1x_3-plane: $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$

Consider the rectangular domain in $\widetilde{\Sigma}_\theta$ on the right of the middle vertical line. It corresponds to a piece of $M_{\theta,\alpha,\frac{\pi}{2}}$ (in fact, we know by S_3 that it is a half of $M_{\theta,\alpha,\frac{\pi}{2}}$), which is a noncompact, singly periodic minimal annulus bounded by four horizontal straight lines. We consider a component $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ of the lifting of this annulus to $\mathbb{R}^3$, and call $S_{\theta,\alpha,\frac{\pi}{2}}$ a fundamental domain of $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ (see Fig. 2).

We can assume that D'' lies at the origin of $\mathbb{R}^3$ and $R_3, \mathcal{D}$ are respectively given by the restrictions to $M_{\theta,\alpha,\frac{\pi}{2}}$ of

$$(x_1, x_2, x_3) \mapsto (x_1 + 1, -x_2, x_3), \qquad (x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3).$$

Take $h > 0$ so that the four horizontal straight lines on the boundary of $S_{\theta,\alpha,\frac{\pi}{2}}$ lie in $\{x_3 = \pm h\}$. Hence both $S_{\theta,\alpha,\frac{\pi}{2}}$ and $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ are contained in the horizontal slab $\{(x_1, x_2, x_3) \mid -h < x_3 < h\}$. Moreover, the horizontal level sections of $S_{\theta,\alpha,\frac{\pi}{2}}$ (which correspond to the vertical lines of $\widetilde{\Sigma}_\theta$ on the right of the middle vertical line)

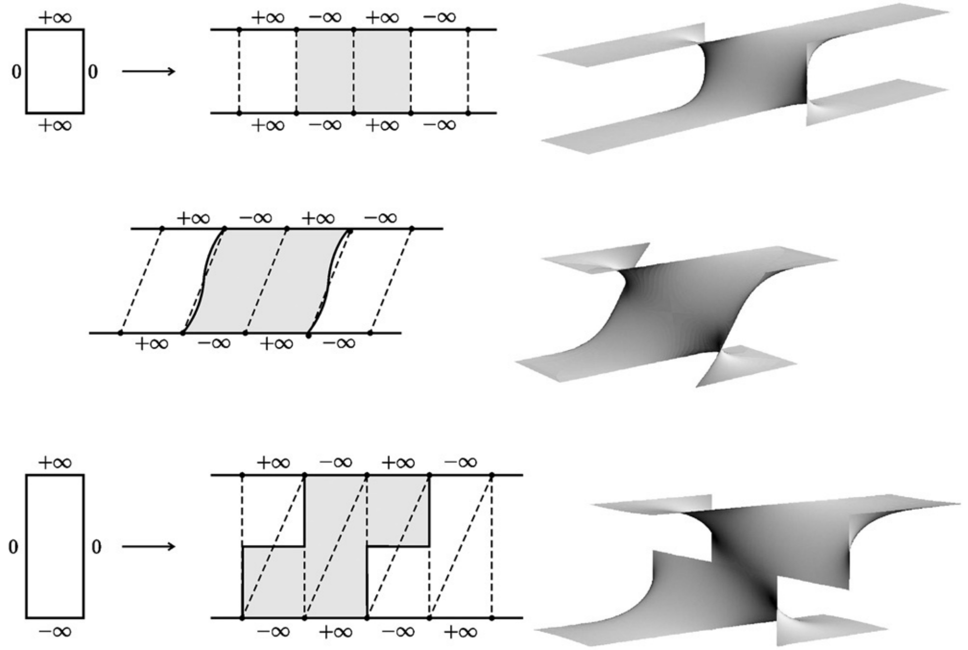

Fig. 2. Construction of the graphs $S_{\frac{\pi}{4},0,\frac{\pi}{2}}$ (top) and $S_{\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{2}}$ (bottom). And the intermediate graph $S_{\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}}$ (center).

have period $P = (2, 0, 0)$, up to sign. Hence $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ projects orthogonally in the x_2-direction onto the whole strip $\mathcal{B} = \{(x_1, 0, x_3) \mid -h < x_3 < h\}$. Finally, let $\Pi : \widetilde{S}_{\theta,\alpha,\frac{\pi}{2}} \to \mathcal{B}$ be the orthogonal projection in the x_2-direction, $\Pi(p) = (x_1(p), 0, x_3(p))$.

Proposition 1. *The surface $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ solves the Jenkins–Serrin problem on $S(h, a)$, for some $a \in (\frac{-1}{2}, \frac{1}{2}]$.*

Proof. Firstly assume $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ is a graph over the strip $\mathcal{B}$, $u : \mathcal{B} \to \widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$. Recall that $M_{\theta,\alpha,\frac{\pi}{2}}$ has horizontal Scherk-type ends with period $(2, 0, 0)$ and that we obtain a fundamental domain of $M_{\theta,\alpha,\frac{\pi}{2}}$ by rotating $S_{\theta,\alpha,\frac{\pi}{2}}$ about one of the four straight lines in $\partial S_{\theta,\alpha,\frac{\pi}{2}}$. Hence the boundary of $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ consists of straight lines whose orthogonal projection in the x_2-direction is formed by two rows of equally spaced points, which we can denote by $p_n = (n - a, 0, -h)$, $q_n = (n + a, 0, h)$, for $n \in \mathbb{Z}$ and some $a \in (\frac{-1}{2}, \frac{1}{2}]$, in such a way that u diverges to $+\infty$ when we approach (p_{2k}, p_{2k+1}), (q_{2k}, q_{2k+1}) and diverges to $-\infty$ when we approach (p_{2k-1}, p_{2k}), (q_{2k-1}, q_{2k}) within $\mathcal{B}$, for every $k \in \mathbb{Z}$. This is, $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ solves the Jenkins–Serrin problem on $S(h, a)$; see Definition 1. Therefore, to conclude Proposition 1 it suffices to prove that $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ is a graph over $\mathcal{B}$.

Denote by $\mathcal{R}$ the piece of $S_{\theta,\alpha,\frac{\pi}{2}}$ which corresponds to the region of $\widetilde{\Sigma}_\theta$ shadowed in Fig. 1; this is, the rectangle of $\widetilde{\Sigma}_\theta$ on the right (resp. left) of the vertical line passing through A, A' (resp. D'', D''') and above (resp. below) the horizontal line passing through D', D'' (resp. D, D'''). The boundary of $\mathcal{R}$ consists of a horizontal curve c_1 in $\mathbb{R}^3$ joining the branch points D'', D''', two curves c_2, c_3 from D'', D''', respectively, to the horizontal plane $\{(x_1, x_2, x_3) \mid x_3 = -h\}$ and either two straight half-lines (when $\alpha \neq 0$) or a straight line (when $\alpha = 0$) in $\{x_3 = -h\}$. Since $R_3(c_2) = c_3$, then $\Pi(c_3) = \Pi(c_2) + (1, 0, 0)$.

Assume $\mathcal{R}$ is a graph over $\mathcal{B}$, and let us prove that the same holds for $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$. Suppose by contradiction there exist two points $p, q \in \widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ with $\Pi(p) = \Pi(q)$. In particular, $x_1(p) = x_1(q)$. Since $\mathcal{D}(x_1, x_2, x_3) = (-x_1, -x_2, -x_3)$ and $\mathcal{R}$ is a graph over $\mathcal{B}$, we can assume $p \in \mathcal{R}$ and $q \in R_3(\mathcal{R})$. Let us call p' the point in c_2 at the same height as p, q (in particular, p', p, q correspond to three points in the same vertical line of $\widetilde{\Sigma}_\theta$). Hence, by using the isometry R_3 and the fact that $\mathcal{R}$ is a graph over $\mathcal{B}$, we deduce $x_1(p') < x_1(p) < x_1(p') + 1 < x_1(q)$, a contradiction.

Therefore, let us prove that $\mathcal{R}$ is a graph over $\mathcal{B}$. The spherical image of $\mathcal{R}$ by its Gauss map is contained in a quarter of a sphere in $\mathbb{S}^2 \cap \{x_2 > 0\}$, so $\mathcal{R}$ is either a graph or a multigraph over $\mathcal{B}$. The following Lemma 1 allows us to conclude that $\mathcal{R}$ cannot be a multigraph, which finishes Proposition 1. $\square$

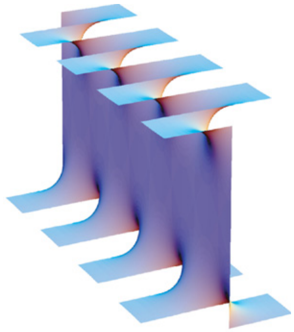

Fig. 3. The Jenkins–Serrin graph $\widetilde{S}_{\frac{\pi}{200},0,\frac{\pi}{2}}$, close to the singly periodic Scherk limit.

Lemma 1. *The restriction of Π to $\sigma = c_3 \cup c_1 \cup c_2$ is one to one.*

Proof. We identify σ with its corresponding curve in Σ_θ. Without loss of generality, we can assume that σ lies in the same branch of the w-map (i.e. w is univalent along σ). Thus we can see z as a parameter on σ, and so $\sigma = \{z = \mathrm{i}t \mid -1 < t < 1\}$. In particular, we can write the first and third coordinates of $M_{\theta,\alpha,\frac{\pi}{2}}$ along σ, denoted by X_1 and X_3 respectively, as functions of t. Since the horizontal level sections of $M_{\theta,\alpha,\frac{\pi}{2}}$ correspond to vertical segments in $\widetilde{\Sigma}_\theta$, it follows that both $X_3|_{c_2}$, $X_3|_{c_3}$ are strictly monotone. Furthermore, the restriction of X_1 to $c_1 = \{z = \mathrm{i}t \mid |t| < \lambda_\theta^{-1}\}$ is also strictly monotone because

$$X_1(t) = \frac{1}{2}\Re \int_{-\mathrm{i}\lambda_\theta^{-1}}^{\mathrm{i}t} \left(\frac{1}{g} - g\right) \mathrm{d}h$$
$$= \mu \int_{-\lambda_\theta^{-1}}^{t} \frac{1 - s^4}{(1 - 2s^2\cos(2\alpha) + s^4)\sqrt{(\lambda_\theta^2 - s^2)(\lambda_\theta^{-2} - s^2)}}\,\mathrm{d}s.$$

Since the Π-projections of c_1, c_2, c_3 are separately embedded and only intersect at the common extrema, we conclude Lemma 1. □

Remark 1. Recall that the period lattice of $M_{\theta,\alpha,\frac{\pi}{2}}$ is generated by $P = (2, 0, 0)$ and $T = (T_1, 0, T_3)$, $T_3 \neq 0$. Then $h = \frac{1}{4}|T_3|$ and $a = \frac{1}{4}|T_1|$ in Proposition 1. In particular, it must hold that $T_1^2 + T_3^2 > 4$.

4. Limit graphs of $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$

We know [7] that $M_{\theta,\alpha,\frac{\pi}{2}}$ converges to two singly periodic Scherk minimal surfaces of angle[1] $\frac{\pi}{2}$ when $\theta \to 0$. Let us recall how we can see the singly periodic Scherk minimal surface of angle $\frac{\pi}{2}$ as a Jenkins–Serrin graph on the half-plane. Consider half a strip $\{0 \le x_1 \le 1,\ x_3 \ge 0\}$, with boundary data 0 on the vertical straight half-lines and $+\infty$ on the unit straight segment in between. By rotating about the boundary half-lines, we obtain a Jenkins–Serrin graph $\widetilde{S}_{1p}$ on the half-plane with boundary values $\pm\infty$ on $\{x_3 = 0\}$ disposed alternately on unitary edges, which is half a singly periodic Scherk minimal surface of angle $\pi/2$ and period $(2, 0, 0)$.
We have proven that $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ is a graph over the marked strip $S(h, a)$, where $h = \frac{1}{4}|T_3|$ and $a = \frac{1}{4}|T_1|$. Translate $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ by $(a, 0, h)$. Then this translated $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ converges to $\widetilde{S}_{1p}$, when $\theta \to 0$ (see Fig. 3). By using the isometry $\mathcal{D}$, we obtain that the translated $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ by $(-a, 0, -h)$ has a similar behavior. In particular, when $\theta \to 0$, the width of the strip diverges to $+\infty$ (i.e. $|T_3| \to +\infty$).

When $\theta \to \frac{\pi}{2}$ and $\alpha \to \alpha_\infty \neq 0$, $M_{\theta,\alpha,\frac{\pi}{2}}$ converges to two doubly periodic Scherk minimal surfaces of angle α_∞ and periods of length 1. Half such a doubly periodic Scherk example can be seen as a Jenkins–Serrin graph $\mathcal{S}_{2p}$ on the corresponding rhombus with alternating boundary data $\pm\infty$.

[1] We define as the *angle* of a singly or doubly periodic Scherk minimal surface the angle between its nonparallel ends.

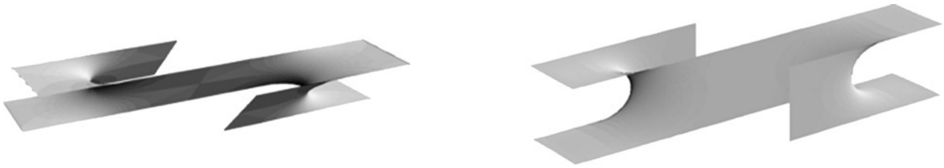

Fig. 4. The Jenkins–Serrin graphs $S_{\frac{49\pi}{100},\frac{\pi}{4},\frac{\pi}{2}}$ (left) and $S_{\frac{49\pi}{100},\frac{\pi}{2},\frac{\pi}{2}}$ (right), close to doubly periodic Scherk minimal surfaces.

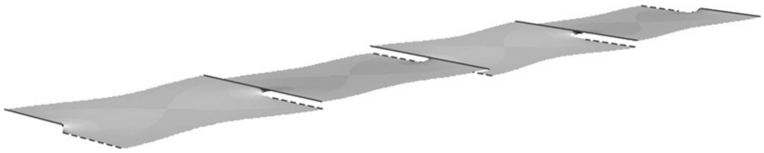

Fig. 5. The Jenkins–Serrin graph $\widetilde{S}_{\frac{49\pi}{100},0,\frac{\pi}{2}}$, close to the helicoid limit.

Denote by $\mathcal{P}_n$ the rhombus of vertices $p_n, p_{n+1}, q_{n+1}, q_n$, for every $n \in \mathbb{Z}$, and let M_n be the piece of $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ over $\mathcal{P}_n$, translated so that $x_2 = 0$ in the middle point of M_n (i.e. the point in M_n which projects onto the middle point of $\mathcal{P}_n$). For any $k \in \mathbb{Z}$, M_{2k} converges to $\mathcal{S}_{2p}$, when $\theta \to \frac{\pi}{2}$ and $\alpha \to \alpha_\infty$ (see Fig. 4); and M_{2k-1} converges to the reflected image of $\mathcal{S}_{2p}$ across the x_1x_3-plane. In this case, $T_1^2 + T_3^2 \to 4$ and $T_3 \not\to 0$. Moreover, for each $T_{1,\infty}, T_{3,\infty}$ with $T_{1,\infty}^2 + T_{3,\infty}^2 = 4$ and $T_{3,\infty} \neq 0$, there exists a $\mathcal{S}_{2p}$ which is graph over the parallelogram determined by $(1, 0, 0)$, $(T_{1,\infty}, 0, T_{3,\infty})$; and this $\mathcal{S}_{2p}$ is obtained as a limit of translated graphs $S_{\theta,\alpha,\frac{\pi}{2}}$.

When $\theta \to \frac{\pi}{2}$ but $\alpha \to 0$, the dilated KMR example $\frac{1}{\mu}M_{\theta,\alpha,\frac{\pi}{2}}$ converges to two vertical helicoids spinning oppositely. Let $\mathcal{H}$ be half a fundamental domain of the vertical helicoid bounded by two horizontal straight lines, both projecting vertically onto the same straight line $\ell \subset \{x_3 = 0\}$. Assume $x_1(\ell) = 0$ and that the projection of $\partial\mathcal{H}$ in the x_2-direction consists of two points at heights $-h$ and h. Thus the interior of $\mathcal{H}$ can be seen as a graph onto the strip $\{(x_1, 0, x_3) \mid -h < x_3 < h\}$, with boundary data $+\infty$ on $\{x_1 > 0, x_2 = 0, x_3 = h\} \cup \{x_1 < 0, x_2 = 0, x_3 = -h\}$, and $-\infty$ on $\{x_1 > 0, x_2 = 0, x_3 = -h\} \cup \{x_1 < 0, x_2 = 0, x_3 = h\}$. As $\theta \to \frac{\pi}{2}$ and $\alpha \to 0$, the suitably translated graphs $\frac{1}{\mu}S_{\theta,\alpha,\frac{\pi}{2}}$ converge to $\mathcal{H}$ (see Fig. 5). And different translations of the surfaces $\frac{1}{\mu}S_{\theta,\alpha,\frac{\pi}{2}}$ converge, when $\theta \to \frac{\pi}{2}$ and $\alpha \to 0$, to another half of a vertical helicoid spinning oppositely. In this case, $T_1^2 + T_3^2 \to 4$ and $T_3 \to 0$.

5. Proof of Theorem 1

Denote by $\mathcal{M}$ the family of graphs

$$\mathcal{M} = \left\{\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}} \mid \theta \in \left(0, \frac{\pi}{2}\right), \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

Recall that we could have defined the graphs $\widetilde{S}_{\theta,\frac{-\pi}{2},\frac{\pi}{2}}$ in a similar way, but $\widetilde{S}_{\theta,\frac{-\pi}{2},\frac{\pi}{2}} = \widetilde{S}_{\theta,\frac{\pi}{2},\frac{\pi}{2}}$. From the classification of the KMR examples [7], we know that no two surfaces in $\mathcal{M}$ coincide. This family $\mathcal{M}$ can be naturally endowed with the product topology given by its parameters (θ, α). Furthermore, we know the surfaces obtained by taking limits from graphs in $\mathcal{M}$ (see Section 4). We deduce that the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ has two components: an isolated point $\{\star\}$ corresponding to the singly periodic Scherk limit $\widetilde{S}_{1p}$, and a closed curve Γ corresponding to the union of the family of doubly periodic Scherk limits and the helicoidal limit (recall that the helicoid can be obtained as a limit surface of doubly periodic Scherk minimal examples). Hence $\mathcal{M}$ is topologically a punctured disk $D - \{\star\}$, where Γ is the boundary of the disk D.

Recall the collection of marked strips defined just after Definition 1,

$$\mathcal{S} = \left\{S(h, a) \mid h > 0 \text{ and } a \in \left(\frac{-1}{2}, \frac{1}{2}\right] \text{ satisfy } a^2 + h^2 > \frac{1}{4}\right\}.$$

Since $S(h, \frac{-1}{2}) = S(h, \frac{1}{2})$, the family S can be topologized by the natural map $S(h, a) \in S \overset{H}{\mapsto} (h, a) \in \mathbb{R}^+ \times (\mathbb{R}/\mathbb{Z})$. Note that the parameter a goes necessarily to 0 when $S(h, a) \in \mathcal{S}$ and $h \to +\infty$. After identifying $\mathcal{S}$ with its image through H, we obtain that $\mathcal{S}$ is topologically a punctured disk $D - \{\star\}$, and the boundary of $\mathcal{S}$ consists of two components: the curve $\{(h, a) \mid h^2 + a^2 = \frac{1}{4}\}$, which corresponds to $\Gamma = \partial D$, and $\{(+\infty, 0)\}$, which corresponds to $\{\star\}$.

Proposition 1 and Remark 1 let us define the continuous map

$$\begin{aligned} \phi : \mathcal{M} \equiv D - \{\star\} \quad &\longrightarrow \quad \mathcal{S} \equiv D - \{\star\}, \\ \widetilde{S}_{\theta,\alpha,\frac{\pi}{2}} \quad &\mapsto \quad S\left(\frac{1}{4}|T_3|, \frac{1}{4}|T_1|\right) \end{aligned}$$

which can be continuously extended to the boundaries so that $\phi(\star) = \star$ and $\phi(\partial D) = \partial D$, using Section 4.

Since the conjugate graph of $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ is contained in $\{(x_1, x_2, x_3) \mid 0 < x_2 < 1\}$, then the following lemma implies that ϕ is injective.

Lemma 2 (*Mazet, [5]*). *Let Ω be a convex polygonal domain with unitary edges, and M be a minimal (vertical) graph on Ω with boundary data $\pm\infty$ disposed alternately, and whose conjugate graph lies on a horizontal slab of width* 1. *Then M is unique up to a vertical translation.*

It is not difficult to obtain that ϕ is onto from the fact that it is continuous, injective and $\phi(\star) = \star$, $\phi(\partial D) = \partial D$. This proves the first part in Theorem 1.

The uniqueness part in Theorem 1 about the graphs $\widetilde{S}_{\theta,\alpha,\frac{\pi}{2}}$ can be also deduced from Lemma 2 as above. This finishes the proof of Theorem 1.

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